

## Qubit Semantics and Quantum Trees

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In the qubit semantics the *meaning* of any sentence  $\alpha$  is represented by a *quregister*: a unit vector of the  $n$ -fold tensor product  $\otimes^n \mathbb{C}^2$ , where  $n$  depends on the number of occurrences of atomic sentences in  $\alpha$  (see Cattaneo *et al.*). The logic characterized by this semantics, called *quantum computational logic* (QCL), is *unsharp*, because the noncontradiction principle is violated. We show that QCL does not admit any logical truth. In this framework, any sentence  $\alpha$  gives rise to a *quantum tree*, consisting of a sequence of unitary operators. The quantum tree of  $\alpha$  can be regarded as a quantum circuit that transforms the quregister associated to the occurrences of atomic subformulas of  $\alpha$  into the quregister associated to  $\alpha$ .

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**KEY WORDS:** quantum computation; quantum logic.

### 1. INTRODUCTION

The theory of logical gates in quantum computation has suggested the semantic characterization of a nonstandard form of quantum logic, that has been called *quantum computational logic*. We will first recall some basic notions of quantum computation. Consider the two-dimensional Hilbert space  $\mathbb{C}^2$  (where any vector  $|\psi\rangle$  is represented by a pair of complex numbers). Let  $\mathcal{B}^{(1)} = \{|0\rangle, |1\rangle\}$  be the canonical orthonormal basis for  $\mathbb{C}^2$ , where  $|0\rangle = (1, 0)$  and  $|1\rangle = (0, 1)$ .

*Definition 1.1.* *Qubit.* A *qubit* is a unit vector  $|\psi\rangle$  of the Hilbert space  $\mathbb{C}^2$ .

Recalling the Born rule, any qubit  $|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle$  (with  $|c_0|^2 + |c_1|^2 = 1$ ), can be regarded as an *uncertain piece of information*, where the answer *NO* has probability  $|c_0|^2$ , while the answer *YES* has probability  $|c_1|^2$ . The two basis-elements  $|0\rangle$  and  $|1\rangle$  are usually taken as encoding the classical bit-values 0

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and 1, respectively. From a semantic point of view, they can be also regarded as the classical truth-values *Falsity* and *Truth*.

An  $n$ -qubit system (also called  $n$ -qregister or quantum register of size  $n$ ) is represented by a unit vector in the  $n$ -fold tensor product Hilbert space  $\otimes^n \mathbb{C}^2 := \underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{n\text{-times}}$ . We will use  $x, y, \dots$  as variables ranging over the set  $\{0, 1\}$ . At the

same time,  $|x\rangle, |y\rangle, \dots$  will range over the basis  $\mathcal{B}^{(1)}$ . Any factorized unit vector  $|x_1\rangle \otimes \dots \otimes |x_n\rangle$  of the space  $\otimes^n \mathbb{C}^2$  will be called an  $n$ -configuration (which can be regarded as a quantum realization of a classical bit sequence of length  $n$ ). Instead of  $|x_1\rangle \otimes \dots \otimes |x_n\rangle$  we will simply write  $|x_1, \dots, x_n\rangle$ . Recall that the dimension of  $\otimes^n \mathbb{C}^2$  is  $2^n$ , while the set of all  $n$ -configurations  $\mathcal{B}^{(n)} = \{|x_1, \dots, x_n\rangle : x_i \in \{0, 1\}\}$  is an orthonormal basis for the space  $\otimes^n \mathbb{C}^2$ . We will call this set a *computational basis* for the  $n$ -qregisters. Since any string  $x_1, \dots, x_n$  represents a natural number  $j \in [0, 2^n - 1]$  (where  $j = 2^{n-1}x_1 + 2^{n-2}x_2 + \dots + x_n$ ), any unit vector of  $\otimes^n \mathbb{C}^2$  can be shortly expressed in the following form:  $\sum_{j=0}^{2^n-1} c_j |j\rangle$ , where  $c_j \in \mathbb{C}$ ,  $|j\rangle$  is the  $n$ -configuration corresponding to the number  $j$  and  $\sum_{j=0}^{2^n-1} |c_j|^2 = 1$ .

## 2. QUANTUM LOGICAL GATES

An  $n$ -input/ $n$ -output *quantum logical gate* is a computation device that transforms an  $n$ -qregister into an  $n$ -qregister. From the mathematical point of view, a quantum logical gate can be described as a unitary operator that acts on the vectors of the Hilbert space  $\otimes^n \mathbb{C}^2$ . We will now introduce some examples of quantum logical gates. Since they are described by unitary operators, it will be sufficient to determine their behavior on the elements of the computational basis  $\mathcal{B}^{(n)}$ .

*Definition 2.1. (The NOT gate).* For any  $n \geq 1$ , the NOT gate is the linear operator  $\text{NOT}^{(n)}$  defined on  $\otimes^n \mathbb{C}^2$  such that for every element  $|x_1, \dots, x_n\rangle$  of the computational basis  $\mathcal{B}^{(n)}$ :

$$\text{NOT}^{(n)}(|x_1, \dots, x_n\rangle) = |x_1, \dots, x_{n-1}\rangle \otimes |1 - x_n\rangle.$$

In other words,  $\text{NOT}^{(n)}$  inverts the value of the last element of any basis-vector of  $\otimes^n \mathbb{C}^2$ .

*Definition 2.2. (The Petri–Toffoli gate).* For any  $n \geq 1$  and any  $m \geq 1$  the Petri–Toffoli gate is the linear operator  $T^{(n,m,1)}$  defined on  $\otimes^{n+m+1} \mathbb{C}^2$  such that for every element  $|x_1, \dots, x_n\rangle \otimes |y_1, \dots, y_m\rangle \otimes |z\rangle$  of the computational basis  $\mathcal{B}^{(n+m+1)}$ :

$$\begin{aligned} T^{(n,m,1)}(|x_1, \dots, x_n\rangle \otimes |y_1, \dots, y_m\rangle \otimes |z\rangle) \\ = |x_1, \dots, x_n\rangle \otimes |y_1, \dots, y_m\rangle \otimes |x_n y_m \oplus z\rangle, \end{aligned}$$

where  $\oplus$  represents the sum modulo 2.

One can easily show that both  $\text{NOT}^{(n)}$  and  $T^{(n,m,1)}$  are unitary operators.

The gate  $T^{(n,m,1)}$  is very similar to a gate introduced by Petri in Petri (1967). For  $n = m = 1$ , we obtain the well known Toffoli gate (Toffoli, 1980), which is essentially identical to Feynman’s *Controlled-Controlled-NOT gate*. Both classical conjunction and classical negation are realized by this gate in a reversible way.

The quantum logical gates we have considered so far are, in a sense, “semiclassical.” A quantum logical behavior only emerges in the case where our gates are applied to superpositions. When restricted to classical registers, such operators turn out to behave as classical truth-functions. We will now consider a *genuine quantum gate* that transforms classical registers (elements of  $\mathcal{B}^{(n)}$ ) into quregisters that are superpositions.

*Definition 2.3. (The square-root-of-NOT gate).* For any  $n \geq 1$ , the square-root-of-NOT is the linear operator  $\sqrt{\text{NOT}}^{(n)}$  defined on  $\otimes^n \mathbb{C}^2$  such that for every element  $|x_1, \dots, x_n\rangle$  of the computational basis  $\mathcal{B}^{(n)}$ :

$$\begin{aligned} \sqrt{\text{NOT}}^{(n)}(|x_1, \dots, x_n\rangle) &= |x_1, \dots, x_{n-1}\rangle \otimes \frac{1}{2}((1+i)|x_n\rangle \\ &\quad + (1-i)|1-x_n\rangle). \end{aligned}$$

One can easily show that  $\sqrt{\text{NOT}}^{(n)}$  is a unitary operator. The basic property of  $\sqrt{\text{NOT}}^{(n)}$  is the following:

$$\text{for any } |\psi\rangle \in \otimes^n \mathbb{C}^2, \quad \sqrt{\text{NOT}}^{(n)}(\sqrt{\text{NOT}}^{(n)}(|\psi\rangle)) = \text{NOT}^{(n)}(|\psi\rangle).$$

In other words, applying twice the square root of the negation means negating.

Interestingly enough, the square-root-of-NOT gate has some physical models and implementations. As an example, consider an idealized atom with a single electron and two energy levels: a *ground state* (identified with  $|0\rangle$ ) and an *excited state* (identified with  $|1\rangle$ ). By shining a pulse of light of appropriate intensity, duration, and wavelength, it is possible to force the electron to change energy level. As a consequence, the state (bit)  $|0\rangle$  is transformed into the state (bit)  $|1\rangle$ , and viceversa:  $|0\rangle \mapsto |1\rangle; \quad |1\rangle \mapsto |0\rangle$ . We have thus obtained a typical physical model for the gate  $\text{NOT}^{(1)}$ .

Now, by using a light pulse of half the duration as the one needed to perform the NOT operation, we effect a half-flip between the two logical states. The state of the atom after the half pulse is neither  $|0\rangle$  nor  $|1\rangle$ , but rather a superposition of both states:  $|0\rangle \mapsto \frac{1+i}{2}|0\rangle + \frac{1-i}{2}|1\rangle; \quad |1\rangle \mapsto \frac{1-i}{2}|0\rangle + \frac{1+i}{2}|1\rangle$ .

As expected, the square-root-of NOT gate has no Boolean counterpart.

**Lemma 2.1.** *There is no function  $f : \{0, 1\} \rightarrow \{0, 1\}$  such that for any  $x \in \{0, 1\} : f(f(x)) = 1 - x$ .*

**Proof:** Suppose, by contradiction, that such a function  $f$  exists. Two cases are possible: (i)  $f(0) = 0$ ; (ii)  $f(0) = 1$ .

(i) By hypothesis,  $f(0) = 0$ . Thus,  $1 = f(f(0)) = f(0) = 0$ , contradiction.

(ii) By hypothesis,  $f(0) = 1$ . Thus,  $1 = f(f(0)) = f(1)$ . Hence,  $f(0) = f(1)$ .

Therefore,  $1 = f(f(0)) = f(f(1)) = 0$ , contradiction. □

Interestingly enough,  $\sqrt{\text{NOT}}$  does not have even any fuzzy counterpart.

**Lemma 2.2.** *There is no continuous function  $f : [0, 1] \rightarrow [0, 1]$  such that for any  $x \in [0, 1] : f(f(x)) = 1 - x$ .*

**Proof:** Suppose, by contradiction, that such a function  $f$  exists. First, we prove that  $f(\frac{1}{2}) = \frac{1}{2}$ . By hypothesis,  $f(f(\frac{1}{2})) = 1 - \frac{1}{2} = \frac{1}{2}$ . Hence,  $f(f(f(\frac{1}{2}))) = f(\frac{1}{2})$ . Thus,  $1 - f(\frac{1}{2}) = f(\frac{1}{2})$ . Therefore,  $f(\frac{1}{2}) = \frac{1}{2}$ . Consider now  $f(0)$ . One can easily show:  $f(0) \neq 0$  and  $f(0) \neq 1$ . Clearly,  $f(0) \neq \frac{1}{2}$  since otherwise we would obtain  $1 = f(f(0)) = f(\frac{1}{2}) = \frac{1}{2}$ . Thus, only two cases are possible:

(i)  $0 < f(0) < \frac{1}{2}$ ; (ii)  $\frac{1}{2} < f(0) < 1$ .

(i) By hypothesis,  $0 < f(0) < \frac{1}{2} < 1 = f(f(0))$ . Consequently, by continuity,  $\exists x \in (0, f(0))$  such that  $\frac{1}{2} = f(x)$ . Accordingly,  $\frac{1}{2} = f(\frac{1}{2}) = f(f(x)) = 1 - x$ . Hence,  $x = \frac{1}{2}$ , which contradicts  $x < f(0) < \frac{1}{2}$ .

(ii) By hypothesis,  $f(\frac{1}{2}) = \frac{1}{2} < f(0) < 1 = f(f(0))$ . By continuity,  $\exists x \in (\frac{1}{2}, f(0))$  such that  $f(x) = f(0)$ . Thus,  $1 - x = f(f(x)) = f(f(0)) = 1$ . Hence,  $x = 0$ , which contradicts  $x > \frac{1}{2}$ . □

Consider now the set  $\bigcup_{n=1}^{\infty} \otimes^n \mathbb{C}^2$  (which contains all quregisters  $|\psi\rangle$  “living” in  $\otimes^n \mathbb{C}^2$ , for a given  $n \geq 1$ ). The gates NOT,  $\sqrt{\text{NOT}}$ , and  $T$  can be uniformly defined on this set in the expected way:

$$\text{NOT}(|\psi\rangle) := \text{NOT}^{(n)}(|\psi\rangle), \quad \text{if } |\psi\rangle \in \otimes^n \mathbb{C}^2$$

$$\sqrt{\text{NOT}}(|\psi\rangle) := \sqrt{\text{NOT}}^{(n)}(|\psi\rangle), \quad \text{if } |\psi\rangle \in \otimes^n \mathbb{C}^2$$

$$T(|\psi\rangle \otimes |\varphi\rangle \otimes |\chi\rangle) := T^{(n,m,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |\chi\rangle),$$

$$\text{if } |\psi\rangle \in \otimes^n \mathbb{C}^2, |\varphi\rangle \in \otimes^m \mathbb{C}^2 \text{ and } |\chi\rangle \in \mathbb{C}^2$$

On this basis, a conjunction AND and a disjunction OR can be defined for any pair of quregisters  $|\psi\rangle$  and  $|\varphi\rangle$ :

$$\text{AND}(|\psi\rangle, |\varphi\rangle) := T(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle).$$

$$\text{OR}(|\psi\rangle, |\varphi\rangle) := \text{NOT}(\text{AND}(\text{NOT}(|\psi\rangle), \text{NOT}(|\varphi\rangle))).$$

Clearly,  $|0\rangle$  represents an “ancilla” in the definition of AND. We will use AND (OR) as a metalinguistic abbreviation for the corresponding *definiens*.

One can easily verify that, when applied to classical bits, NOT, AND, and OR behave as the standard Boolean truth-functions.

We will now introduce the concept of *probability-value of a quregister*, which will play an important role in the quantum computational semantics. For any integer  $n \geq 1$ , let us first define a particular set of natural numbers:

$$C_1^{(n)} := \{i : \|i\rangle = |x_1, \dots, x_n\rangle, \text{ and } x_n = 1\}.$$

Apparently,  $C_1^{(n)}$  contains precisely all the odd numbers in  $[0, 2^n - 1]$ .

*Definition 2.4. (Probability-value).* Let  $|\psi\rangle = \sum_{j=0}^{2^n-1} c_j \|j\rangle$  be any quregister of  $\otimes^n \mathbb{C}^2$ . The *probability-value of  $|\psi\rangle$*  is the real value  $\text{Prob}(|\psi\rangle) := \sum_{j \in C_1^{(n)}} |c_j|^2$ .

From an intuitive point of view,  $\text{Prob}(|\psi\rangle)$  represents the probability that the quregister  $|\psi\rangle$  (which is a superposition) *collapses* into an  $n$ -configuration whose last element is 1.

**Theorem 2.1.** *Let  $|\psi\rangle$  and  $|\varphi\rangle$  be two quregisters. The following properties hold:*

- (i)  $\text{Prob}(\text{AND}(|\psi\rangle, |\varphi\rangle)) = \text{Prob}(|\psi\rangle)\text{Prob}(|\varphi\rangle)$ ;
- (ii)  $\text{Prob}(\text{NOT}(|\psi\rangle)) = 1 - \text{Prob}(|\psi\rangle)$ ;
- (iii)  $\text{Prob}(\text{OR}(|\psi\rangle, |\varphi\rangle)) = \text{Prob}(|\psi\rangle) + \text{Prob}(|\varphi\rangle) - \text{Prob}(|\psi\rangle)\text{Prob}(|\varphi\rangle)$ ;
- (iv) *Let  $|\psi\rangle = \sum_{j=0}^{2^n-1} a_j \|j\rangle$ . Then*  
 $\text{Prob}(\sqrt{\text{NOT}}(|\psi\rangle)) = \sum_{j \in C_1^{(n)}} \left| \frac{1-i}{2} a_{j-1} + \frac{1+i}{2} a_j \right|^2$ ;
- (v)  $\text{Prob}(\sqrt{\text{NOT}}(\text{NOT}(|\psi\rangle))) = \text{Prob}(\text{NOT}(\sqrt{\text{NOT}}(|\psi\rangle)))$ ;
- (vi) *Let  $|\psi\rangle = \sum_{j=0}^{2^n-1} a_j \|j\rangle |x_j\rangle$ . Then  $\text{Prob}(\sqrt{\text{NOT}}(|\psi\rangle)) = \frac{1}{2}$ ;*
- (vii)  $\text{Prob}(\sqrt{\text{NOT}}(\text{AND}(|\psi\rangle, |\varphi\rangle))) = \frac{1}{2}$ .

**Proof:**

(i)–(v) (*Dalla chiara et al.*);

$$\begin{aligned} \text{(vi) } \text{Prob}(\sqrt{\text{NOT}}(|\psi\rangle)) &= \text{Prob} \left( \sqrt{\text{NOT}} \left( \sum_{j=0}^{2^n-1} a_j \|j\rangle |x_j\rangle \right) \right) \\ &= \text{Prob} \left( \sum_{j=0}^{2^n-1} a_j \|j\rangle \otimes \left( \frac{1}{2}(1+i)|x_j\rangle + \frac{1}{2}(1-i)|1-x_j\rangle \right) \right) \end{aligned}$$

$$\begin{aligned}
 &= \text{Prob} \left( \sum_{j=0}^{2^n-1} a_j \frac{1}{2} (1 - i(-1)^{x_j}) \|j\rangle \right) |1\rangle \\
 &\quad + \sum_{j=0}^{2^n-1} a_j \frac{1}{2} (1 + i(-1)^{x_j}) \|j\rangle |0\rangle \Big) \\
 &= \sum_{j=0}^{2^n-1} \left| a_j \frac{1}{2} (1 - i(-1)^{x_j}) \right|^2 = \sum_{j=0}^{2^n-1} |a_j|^2 \left| \frac{1}{2} (1 - i(-1)^{x_j}) \right|^2 \\
 &= \frac{1}{2} \sum_{j=0}^{2^n-1} |a_j|^2 = \frac{1}{2};
 \end{aligned}$$

(vii) AND ( $|\psi\rangle, |\varphi\rangle$ ) has the form  $\sum_{j=0}^{2^{n+m}-1} a_j \|j\rangle |x_j\rangle$ .

Thus, by (vi),  $\text{Prob}(\sqrt{\text{NOT}}(\text{AND}(|\psi\rangle, |\varphi\rangle))) = \frac{1}{2}$ . □

### 3. QUANTUM COMPUTATIONAL SEMANTICS

The starting point of the quantum computational semantics is quite different from the standard quantum logical approach. The basic idea is that every sentence  $\alpha$  is semantically interpreted as a quregister. From an intuitive point of view, one can say that the meaning of a sentence is identified with the *information quantity* encoded by the sentence under consideration.

Consider a sentential language  $\mathcal{L}$  with the following connectives: *negation* ( $\neg$ ), *square root of not* ( $\sqrt{\neg}$ ), *conjunction* ( $\wedge$ ). Let  $\text{Form}^{\mathcal{L}}$  be the class of all sentences of the language  $\mathcal{L}$ . We will use the following metavariables:  $p, q, \dots$  for atomic sentences and  $\alpha, \beta, \dots$  for sentences.

The basic concept of our semantics is represented by the notion of *quantum computational model*: an interpretation of the language  $\mathcal{L}$  that associates a quregister to any sentence  $\alpha$ .

*Definition 3.1.:* (Quantum computational model). A *quantum computational model* of  $\mathcal{L}$  is a function  $Qub : \text{Form}^{\mathcal{L}} \rightarrow \bigcup_{n=1}^{\infty} \otimes^n \mathbb{C}^2$  that associates to any sentence  $\alpha$  of the language a quregister:

$$Qub(\alpha) := \begin{cases} \text{a qubit} & \text{if } \alpha \text{ is an atomic sentence;} \\ \text{NOT}(Qub(\beta)) & \text{if } \alpha = \neg\beta; \\ \sqrt{\text{NOT}}(Qub(\beta)) & \text{if } \alpha = \sqrt{\neg}\beta; \\ \text{AND}(Qub(\beta), Qub(\gamma)) & \text{if } \alpha = \beta \wedge \gamma. \end{cases}$$

We will call  $Qub(\alpha)$  the *information-value* of  $\alpha$ . Instead of  $Qub(\alpha)$ , we will also write  $|\alpha\rangle_{Qub}$  (or simply  $|\alpha\rangle$ ). Our definition univocally determines, for any  $Qub$  and any sentence  $\alpha$ , the Hilbert space  $\otimes^n \mathbb{C}^2$  to which  $|\alpha\rangle_{Qub}$  belongs. Apparently,  $n$  is the number of all occurrences of atomic sentences and of the connective  $\wedge$  in  $\alpha$ . According to the intended physical interpretation,  $Qub$  will associate to each occurrence of one and the same atomic subformula  $p$  of  $\alpha$  the state  $|p\rangle$ , that corresponds to an *identical preparation* of the quantum system.

We can now define the notion of *truth*, *logical truth*, *consequence*, and *logical consequence*.

*Definition 3.2. (Truth and logical truth).* A sentence  $\alpha$  is *true* in a quantum computational model  $Qub$  (abbreviated as  $\models_{Qub} \alpha$ ) iff  $\text{Prob}(Qub(\alpha)) = 1$ ;  $\alpha$  is a *logical truth* ( $\models \alpha$ ) iff for any  $Qub$ ,  $\models_{Qub} \alpha$ .

*Definition 3.3. (Consequence in Qub and logical consequence).* A sentence  $\beta$  is a *consequence* in a quantum computational model  $Qub$  of a sentence  $\alpha$  ( $\alpha \models_{Qub} \beta$ ) iff  $\text{Prob}(Qub(\alpha)) \leq \text{Prob}(Qub(\beta))$ ;  $\beta$  is a *logical consequence* of  $\alpha$  ( $\alpha \models \beta$ ) iff for any  $Qub$ ,  $\alpha \models_{Qub} \beta$ .

The logic characterized by this semantics has been termed *quantum computational logic* (QCL, for short)(Cattaneo *et al.*). The following theorem shows that this logic is completely different from the well known *orthomodular quantum logic* (OQL), which is semantically characterized by the class of all orthomodular lattices.

**Theorem 3.1.** *QCL and OQL are not comparable.*

**Proof:** (i) OQL is not a sublogic of QCL. This follows from the fact that the idempotence property ( $\alpha \models \alpha \wedge \alpha$ ) holds in OQL, whereas it is violated in QCL. Take for example,  $|\alpha\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ . Then,  $\text{Prob}(|\alpha\rangle) = \frac{1}{2} > \frac{1}{4} = \text{Prob}(|\alpha \wedge \alpha\rangle)$ . (ii) QCL is not a sublogic of OQL. This follows from the fact that the strong distributivity property ( $\alpha \wedge (\beta \vee \gamma) \models (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ ) is violated in OQL (Dalla Chiara and Giuntini (2002)), whereas it holds in QCL. In fact, by Theorem 1 (i)–iii), we obtain

$$\begin{aligned} \text{Prob}(|\alpha \wedge (\beta \vee \gamma)\rangle) &= \text{Prob}(\text{AND}(|\alpha\rangle, \text{OR}(|\beta\rangle, |\gamma\rangle))) \\ &= \text{Prob}(|\alpha\rangle)\text{Prob}(|\beta\rangle) + \text{Prob}(|\alpha\rangle)\text{Prob}(|\gamma\rangle) - \text{Prob}(|\alpha\rangle)\text{Prob}(|\beta\rangle) \\ &\quad \times \text{Prob}(|\gamma\rangle) \leq \text{Prob}(|\alpha\rangle)\text{Prob}(|\beta\rangle) + \text{Prob}(|\alpha\rangle)\text{Prob}(|\gamma\rangle) \\ &\quad - \text{Prob}(|\alpha\rangle)^2\text{Prob}(|\beta\rangle)\text{Prob}(|\gamma\rangle) \end{aligned}$$

$$\begin{aligned}
&= \text{Prob}(\text{OR}(\text{AND}(|\alpha\rangle, |\beta\rangle), \text{AND}(|\alpha\rangle, |\gamma\rangle))) \\
&= \text{Prob}(|(\alpha \wedge \beta)\rangle \vee (\alpha \wedge \gamma)|)
\end{aligned}$$

□

The logic QCL turns out to be *unsharp*, because the noncontradiction principle can be violated: the negation of a contradiction ( $\neg(\alpha \wedge \neg\alpha)$ ) is not necessarily true (Cattaneo *et al.*).

**Theorem 3.2.** *Let  $Qub$  be any quantum computational model and let  $\alpha$  be any sentence. If  $\text{Prob}(Qub(\alpha)) \in \{0, 1\}$ , then there is an atomic subformula  $p$  of  $\alpha$  such that  $\text{Prob}(Qub(p)) \in \{0, \frac{1}{2}, 1\}$ .*

**Proof:** Suppose that  $\text{Prob}(Qub(\alpha)) \in \{0, 1\}$ . The proof is by induction on the logical complexity of  $\alpha$ .

- (i)  $\alpha$  is an atomic sentence. The proof is trivial.
- (ii)  $\alpha = \neg\beta$ . By Theorem 2.1(ii),  $\text{Prob}(Qub(\alpha)) = 1 - \text{Prob}(Qub(\beta)) \in \{0, 1\}$ . The conclusion follows by induction hypothesis.
- (iii)  $\alpha = \sqrt{-}\beta$ . By hypothesis and by Theorem 2.1(vii),  $\beta$  cannot be a conjunction. Consequently, only the following cases are possible: (iiia)  $\beta = p$ ; (iiib)  $\beta = \neg\gamma$ ; (iiic)  $\beta = \sqrt{-}\gamma$ .
- (iiia)  $\beta = p$ . By hypothesis,  $\text{Prob}(\sqrt{-}\beta) \in \{0, 1\}$ . Hence,  $\sqrt{\text{NOT}}(Qub(p)) = c|x\rangle$ , where  $|x\rangle \in \{|0\rangle, |1\rangle\}$  and  $|c| = 1$ . We have:  $\text{NOT}(Qub(p)) = \sqrt{\text{NOT}}(\sqrt{\text{NOT}}(Qub(p))) = \sqrt{\text{NOT}}(c|x\rangle)$ . By Theorem 2.1(iv),  $\text{Prob}(\sqrt{\text{NOT}}(c|x\rangle)) = \frac{1}{2}$ . As a consequence,  $\text{Prob}(Qub(\neg p)) = \frac{1}{2} = \text{Prob}(Qub(p))$ .
- (iiib)  $\beta = \neg\gamma$ . By Theorem 2.1(v),  $\text{Prob}(Qub(\sqrt{-}\neg\gamma)) = \text{Prob}(Qub(\neg\sqrt{-}\gamma)) = 1 - \text{Prob}(Qub(\sqrt{-}\gamma))$ . The conclusion follows by induction hypothesis.
- (iiic)  $\beta = \sqrt{-}\gamma$ . Then  $\text{Prob}(Qub(\sqrt{-}\sqrt{-}\gamma)) = \text{Prob}(Qub(\neg\gamma)) = 1 - \text{Prob}(Qub(\gamma))$ . The conclusion follows by induction hypothesis.
- (iv)  $\alpha = \beta \wedge \gamma$ . By Theorem 2.1(i),  $\text{Prob}(Qub(\beta \wedge \gamma)) = \text{Prob}(Qub(\beta)) \text{Prob}(Qub(\gamma)) \in \{0, 1\}$ . The conclusion follows by induction hypothesis.

□

A remarkable property of QCL is asserted by the following Corollary of Theorem 3.2.

**Corollary 3.1.** *There exists no quantum computational logical truth.*



**Proof:** Suppose, by contradiction, that  $\alpha$  is a logical truth. Let  $p_1, \dots, p_n$  be the atomic sentences occurring in  $\alpha$  and let  $Qub$  be a quantum computational model such that for any  $i$  ( $1 \leq i \leq n$ ),  $\text{Prob}(Qub(p_i)) \notin \{0, \frac{1}{2}, 1\}$ . Then, by Theorem 3.2,  $\text{Prob}(Qub(\alpha)) \notin \{0, 1\}$ , contradiction.  $\square$

#### 4. QUANTUM TREES

For the sake of technical simplicity we slightly modify our language. The new language contains a privileged atomic sentence  $f$  (representing *the falsity*) and three primitive connectives: the negation  $\neg$ , the square root of the negation  $\sqrt{\neg}$ , and a ternary conjunction  $\bigwedge$ . The connective  $\bigwedge$  represents a conjunction whose form is “close” to the Petri–Toffoli gate. For any sentences  $\alpha$  and  $\beta$  the expression  $\bigwedge(\alpha, \beta, f)$  is a sentence of the language. The usual conjunction  $\alpha \wedge \beta$  is dealt with as metalinguistic abbreviation for the ternary conjunction  $\bigwedge(\alpha, \beta, f)$ . Semantically, we will require that for any  $Qub$ :

$$Qub(f) = |0\rangle; \quad Qub(\bigwedge(\alpha, \beta, f)) = T(Qub(\alpha) \otimes Qub(\beta) \otimes Qub(f)).$$

*Definition 4.1.* (The Atomic Complexity of  $\alpha$ ). The atomic complexity of a sentence  $\alpha$  ( $Atcompl(\alpha)$ ) is the number of occurrences of atomic sentences in  $\alpha$ .

For example, if  $\alpha = p \wedge \neg p = \bigwedge(p, \neg p, f)$ , then  $Atcompl(\alpha) = 3$ .

**Lemma 4.1.** Let  $Atcompl(\alpha) = n$ . Then  $\forall Qub : Qub(\alpha) \in \otimes^n \mathbb{C}^2$ .

Hence, the space of all possible qubit-meanings of  $\alpha$  is determined by the atomic complexity of  $\alpha$ .

We will first introduce the notion of *syntactical tree* of a sentence  $\alpha$  (abbreviated as  $STree^\alpha$ ). Consider all subformulas of  $\alpha$ .

Any subformula may be:

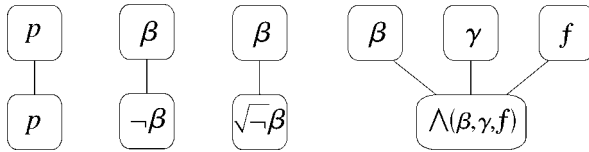
- an *atomic* sentence  $p$  (possibly  $f$ );
- a *negated* sentence  $\neg\beta$ ;
- a *square-root-negated* sentence  $\sqrt{\neg}\beta$ ;
- a *conjunction*  $\bigwedge(\beta, \gamma, f)$ .

The intuitive idea of *syntactical tree* can be illustrated as follows. Every occurrence of a subformula of  $\alpha$  gives rise to a *node* of  $STree^\alpha$ . The tree consists of a finite number of *levels* and each level is represented by a sequence of subformulas

of  $\alpha$ :

$$\begin{aligned} &Level_k(\alpha) \\ &\vdots \\ &Level_1(\alpha). \end{aligned}$$

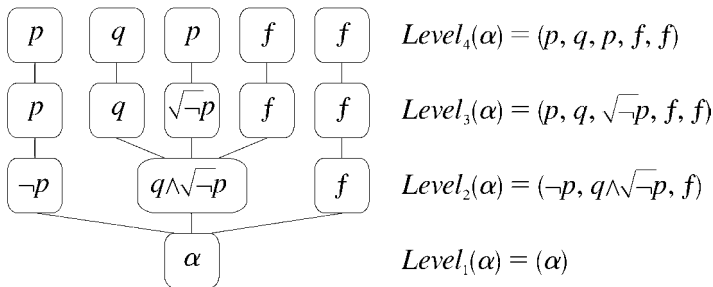
The *root-level* (denoted by  $Level_1(\alpha)$ ) consists of  $\alpha$ . From each node of the tree at most three edges may branch according to the following *branching-rule*:



The second level ( $Level_2(\alpha)$ ) is the sequence of subformulas of  $\alpha$  that is obtained by applying the branching-rule to  $\alpha$ . The third level ( $Level_3(\alpha)$ ) is obtained by applying the branching-rule to each element (node) of  $Level_2(\alpha)$ , and so on. Finally, one obtains a level represented by the sequence of all atomic occurrences of  $\alpha$ . This represents the *last level* of  $STree^\alpha$ . The *height* of  $STree^\alpha$  (denoted by  $Height(\alpha)$ ) is then defined as the number of levels of  $STree^\alpha$ .

A more formal definition of *syntactical tree* can be given by using some standard graph-theoretical notions.

*Example 4.1.* The syntactical tree of  $\alpha = \neg p \wedge (q \wedge \sqrt{\neg}p)$  is the following:



Clearly the height of  $STree^\alpha$  is 4.

For any choice of a quantum computational model  $Qub$ , the syntactical tree of  $\alpha$  determines a corresponding sequence of quregisters. Consider a sentence  $\alpha$  with  $n$  atomic occurrences  $(p_1, \dots, p_n)$ . Then  $Qub(\alpha) \in \otimes^n \mathbb{C}^2$ . We can associate

a quregister  $|\psi_i\rangle$  to each  $Level_i(\alpha)$  of  $Stree^\alpha$  in the following way. Suppose that:

$$Level_i(\alpha) = (\beta_1, \dots, \beta_r).$$

Then:

$$|\psi_i\rangle = Qub(\beta_1) \otimes \dots \otimes Qub(\beta_r)$$

Hence:

$$\left\{ \begin{array}{l} |\psi_1\rangle = Qub(\alpha) \\ \vdots \\ |\psi_{Height(\alpha)}\rangle = Qub(p_1) \otimes \dots \otimes Qub(p_n) \end{array} \right.$$

where all  $|\psi_i\rangle$  belong to the same space  $\otimes^n \mathbb{C}^2$ .

From an intuitive point of view,  $|\psi_{Height(\alpha)}\rangle$  can be regarded as a kind of *epistemic state*, corresponding to the input of a computation, while  $|\psi_1\rangle$  represents the output.

We obtain the following correspondence:

$$\begin{aligned} Level_{Height(\hat{\alpha})}(\alpha) &\leftrightarrow |\psi_{Height(\alpha)}\rangle : \text{the input} \\ &\dots \leftrightarrow \dots \\ Level_1(\alpha) &\leftrightarrow |\psi_1\rangle : \text{the output} \end{aligned}$$

The notion of *quantum tree* of a sentence  $\alpha$  ( $QTree^\alpha$ ) can be now defined as a particular sequence of unitary operators that is uniquely determined by the syntactical tree of  $\alpha$ . As we already know, each  $Level_i(\alpha)$  of  $STree^\alpha$  is a sequence of subformulas of  $\alpha$ . Let  $Level_i^j(\alpha)$  represent the  $j$ -th element of  $Level_i(\alpha)$ . Each node  $Level_i^j(\alpha)$  (where  $1 \leq i < Height(\alpha)$ ) can be naturally associated to a unitary operator  $Op_i^j$ , according to the following *operator-rule*:

$$Op_i^j := \begin{cases} 0^{(1)} & \text{if } Level_i^j(\alpha) \text{ is an atomic sentence;} \\ NOT^{(r)} & \text{if } Level_i^j(\alpha) = \neg\beta \text{ and } |\beta\rangle \in \otimes^r \mathbb{C}^2; \\ \sqrt{NOT}^{(r)} & \text{if } Level_i^j(\alpha) = \sqrt{\neg}\beta \text{ and } |\beta\rangle \in \otimes^r \mathbb{C}^2; \\ T^{(r,s,1)} & \text{if } Level_i^j(\alpha) = \bigwedge(\beta, \gamma, f), |\beta\rangle \in \otimes^r \mathbb{C}^2 \\ & \text{and } |\gamma\rangle \in \otimes^s \mathbb{C}^2, \end{cases}$$

where  $\mathbf{I}^{(1)}$  is the identity operator on  $\mathbb{C}^2$ .

On this basis, one can associate an operator  $U_i$  to each  $Level_i(\alpha)$  (such that  $1 \leq i < Height(\alpha)$ ):

$$U_i := \bigotimes_{j=1}^{|Level_i(\alpha)|} Op_i^j,$$

where  $|Level_i(\alpha)|$  is the length of the sequence  $Level_i(\alpha)$ .

Being the tensor product of unitary operators, every  $U_i$  turns out to be a unitary operator. One can easily show that all  $U_i$  are defined in the same space  $\otimes^n \mathbb{C}^2$ , where  $n$  is the atomic complexity of  $\alpha$ .

The notion of *quantum tree* of a sentence can be now defined as follows.

*Definition 4.2.* (The quantum tree of  $\alpha$ ). The *quantum tree* of  $\alpha$  (denoted by  $QTree^\alpha$ ) is the operator-sequence  $(U_1, \dots, U_{Height(\alpha)-1})$  that is uniquely determined by the syntactical tree of  $\alpha$ .

As an example, consider the following sentence:  $\alpha = p \wedge \neg p = \bigwedge(p, \neg p, f)$ . The syntactical tree of  $\alpha$  is the following:

$$Level_1(\alpha) = \bigwedge(p, \neg p, f)$$

$$Level_2(\alpha) = (p, \neg p, f)$$

$$Level_3(\alpha) = (p, p, f).$$

In order to construct the quantum tree of  $\alpha$ , let us first determine the operators  $Op_i^j$  corresponding to each node of  $Stree^\alpha$ . We will obtain:

- $Op_1^1 = T^{(1,1,1)}$ , because  $\bigwedge(p, \neg p, f)$  is connected with  $(p, \neg p, f)$  (at  $Level_2(\alpha)$ );
- $Op_2^1 = \mathbb{I}^{(1)}$ , because  $p$  is connected with  $p$  (at  $Level_3(\alpha)$ );
- $Op_2^2 = NOT^{(1)}$ , because  $\neg p$  is connected with  $p$  (at  $Level_3(\alpha)$ );
- $Op_2^3 = \mathbb{I}^{(1)}$ , because  $f$  is connected with  $f$  (at  $Level_3(\alpha)$ ).

The quantum tree of  $\alpha$  is represented by the operator-sequence  $(U_1, U_2)$ , where:

$$U_1 = Op_1^1 = T^{(1,1,1)};$$

$$U_2 = Op_2^1 \otimes Op_2^2 \otimes Op_2^3 = \mathbb{I}^{(1)} \otimes NOT^{(1)} \otimes \mathbb{I}^{(1)}.$$

Apparently,  $QTree^\alpha$  is independent of the choice of Qub.

**Theorem 4.1.** Let  $\alpha$  be a sentence whose quantum tree is the operator-sequence  $(U_1, \dots, U_{Height(\alpha)-1})$ . Given a quantum computational model Qub, consider the quregister-sequence  $(|\psi_1\rangle, \dots, |\psi_{Height(\alpha)}\rangle)$  that is determined by Qub and by the syntactical tree of  $\alpha$ . Then,  $U_i(|\psi_{i+1}\rangle) = |\psi_i\rangle$  (for any  $i$  such that  $1 \leq i < Height(\alpha)$ ).

**Proof:** Straightforward, □

The quantum tree of  $\alpha$  can be naturally regarded as a *quantum circuit* that computes the output  $Qub(\alpha)$ , given the input  $Qub(p_1), \dots, Qub(p_n)$  (where  $p_1, \dots, p_n$  are the atomic occurrences of  $\alpha$ ). In this framework, each  $U_i$  is the unitary operator that describes the computation performed by the  $i$ th layer of the circuit.

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